Life of Fred®
Numerical Analysis

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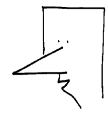
A Note Before We Begin



If this is your first venture into the *Life of Fred* series, a little introduction is needed. This is Fred. He has been teaching at KITTENS University for about six years now. Before this semester he had taught all the undergraduate courses in the math department except numerical analysis.

His teaching style has made him internationally famous. Students from around the world have flocked to KITTENS so that they could experience the human touch that Fred offers. This will all become clear as you read Chapter one.

Math majors at most major universities require a course in numerical analysis. In all the other math courses, you found exact answers. In calculus you learned that the derivative of x^6 was $6x^5$. You learned that the antiderivative of $\sec^2 x$ was $\tan x + C$.

You were taught a half dozen different approaches to finding the exact value of $\int f(x)\,dx$. What you weren't taught was that you can only find the definite integral of less than 1% of all functions. With numerical analysis you will find the value of $\int_{x=0}^{1} \frac{1}{1+x^3}\,dx$, which no calculus student can do.

In algebra you were taught how to solve linear equations and quadratic equations. But what about quintic (5^{th} power) equations?

Or how about solving $x^x = 5$?

The good news is that numerical analysis will allow you to solve virtually all of these problems. You will be able to solve most second-order differential equations, not just the special cases that you learned in calculus.

The bad news is that the answers you get in numerical analysis are only approximations.

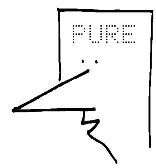
The good news is that these approximations will be given to you with as many decimal places as you desire. If you are working in the **muddy** world of *things*, you don't need 100 decimal places in your answer. Do cabinet makers work with tolerances of a hundred of an inch? Probably not. Do bankers need more than four or five decimal places to play with money? I hope not. Do physicists need more than a dozen decimal places in their answers? If they do, you can give them 40 decimal places.

In this world of applied mathematics, you will be able to work with a zillion more kinds of problems than you ever could in pure math. A whole new world will open up.

Contents

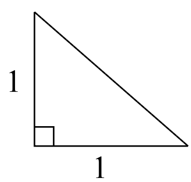
Chapter One
Pure Mathematics

Fred lives in room 314 on the third floor of the Math Building on the KITTENS University campus. He has exactly one doll, Kingie.* The story of how Fred began teaching at KITTENS at the age of nine months is told in *Life of Fred: Calculus Expanded Edition.* Fred is now six years old.



Fred is a pure mathematician. He has taught only pure mathematics: arithmetic,
algebra,
geometry,
trig,
calculus,
logic,
set theory, and so on.

Things are solid. What is, is. What isn't, isn't. $x - x$ always equals zero. His room number is 314 *exactly*. Not 314.15926. He has *exactly* one doll.



In geometry if you have a right triangle with the legs both equal to one, then the hypotenuse must exactly equal $\sqrt{2}$.

In logic if you know P is true and you know that $P \Rightarrow Q$, then Q must be true.

In set theory if set A = {✐, ✈, ☎}, then the cardinality of A must be 3, not 2.98. (Cardinality is the number of elements in a set.)

In trig sin 30º = 0.5. The side opposite a 30º angle in a right triangle is exactly half the length of the hypotenuse.

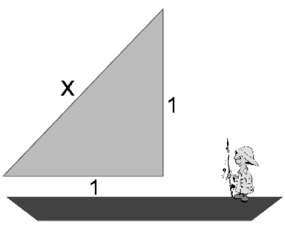
In algebra when Fred factored $10x^2 + 13x - 3$ into $(2x + 3)(5x - 1)$ he felt that his world was as clear as a flawless diamond.

* Kingie is pronounced KING-ee.

In short, Fred was happy in his world of pure mathematics.

A knock on his office door. It was Joe, one of his students. Joe always had a different view of math. This may sound strange, but Joe wanted to *use* Fred's pure mathematics.

"I have a question," Joe began. He always started that way. He figured that if he said that, his listener would have to pay more attention to his words. Given his prolixity* he needed all the help he could get. "As you know, I like to go fishing. I measured it. The height of my mast is equal to 1** And the length from the front of my boat (what everyone else calls the bow) to the foot of the mast is also 1. Let the length of the wire from the front of the boat to the top of the mast be equal to x."



Joe's drawing

Fred nodded and mentally computed that the length of the wire, x, must be equal to $\sqrt{2}$.

Joe had forgotten about the Pythagorean theorem. He just measured the wire and found that it was 1.4. He announced to Fred that x = 1.4

But $\sqrt{2} \neq 1.4$. You can check that by computing 1.4^2, which equals 1.96.

So x – x doesn't equal zero in this case. Joe's x ≠ Fred's x.

The problem was that Fred was living in the crystal world of pure math and Joe was in the **muddy** world of *things*.

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* Some people call him a wordy-birdie. Others say he's a blabbermouth.

** I'm omitting the units. Then I don't have to get in the controversy over metric vs. the imperial system. If you absolutely must need to know, Joe's mast is 1 dekameter tall. (= 10 meters) In British English it's *deca*.

When Joe's girlfriend Darlene measured the wire, she found that $x = 1.4142$. Her eyes were much better than Joe's. She also lives in the **muddy** world of *things*. And 1.4142^2 equals 1.9999616, but this is also far short of 2.

In the **muddy** world of *things*, it is common that $x - x \neq 0$.

Joe explained to Fred (gasp!*) that x is actually about 1.5, since you needed a little extra wire to tie each end.

Here are two major facts about the world of pure math and the **muddy** world of *things*:

Fact #1 Fred's pure math is unreasonably effective in the **muddy** world of *things*.

Why should the things we compute in our heads have any relationship to what happens out there in the everyday world?

Fred taught his arithmetic students that if you wait 2 minutes and then wait 3 more minutes, you will have waited a total of 5 minutes.** When you travel 3 feet/sec for 5 seconds, you will go 15 feet. $d = rt$ When you do calculus and compute the area under one arc of $y = \sin x$, you get 2. When you draw the graph and make little squares and measure the area, it turns out to be pretty close to 2 depending on how accurately you draw your graph.

> You don't have to read any of these "Asides" in this book. I'm writing them for my entertainment or to include review material or to solve messy equations that no human should have to read.
>
> *An Aside*
>
> $$\int_{x=0}^{\pi} \sin x \, dx = -\cos x \Big]_0^{\pi} = 1 + 1 = 2$$

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* I never thought I'd ever write those four words. Would you like to explain physics to Einstein or music to Mozart?

** Seconds, minutes, and hours are in both the metric and imperial systems.

But why does it work? There is no law that what we think should have some correspondence to the **muddy** world of *things*.

In economics, sociology, or political science, there is often strong disagreement—opposing theories about the **muddy** world of *things*.

In mathematics there is much more peace. Where math intersects with the outside world, it is usually pretty easy to check whether 2 + 2 equals 4 or equals 5.

Fact #2 The world of pure math is a teeny tiny bit of all of reality. All the stuff you learned from Fred in algebra, geometry, and calculus is virtually never encountered in real life.

> *An Aside*
>
> Fred would say that all the things he has taught are "real life." They are the things that are eternally true. They are the things that can be relied on.
>
> It is the things in the muddy world that are transitory. This muddy world is the Shadowlands, to steal C.S. Lewis's word.

In geometry: Have you ever seen a circle in the **muddy** world of *things*? Is a pizza ever a perfect circle?

Have you ever encountered an isosceles triangle? If one side is exactly 5.38, what are the chances that a second side will be exactly 5.38? The chances are less than 0.0000000000000000000000000000000001%.

The second side might be 5.38000000000007 or
5.3800000000000006 or
5.3800000000003987 or
5.38000000000000000000000076.

One fun exercise is to list a billion numbers that are between 5.38 and 5.381. One way that you might not have seen before is:

5.3801
5.38012
5.380123
5.3801234
. . .
5.38012345678
5.380123456789
5.38012345678910
5.3801234567891011
5.380123456789101112
. . .
5.380123456789101112131415 . . .98
5.380123456789101112131415 . . .98 99
5.380123456789101112131415 . . .98 99100   and so on.

In algebra: Fred taught you how to factor $ax^2 + bx + c$ where a, b, and c are integers. (In symbols: $a, b, c \in \mathbb{Z}$.)

He never mentioned that virtually no quadratic polynomial with integer coefficents is factorable. Viz.* if a, b, and c are selected at random from $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$ the chances that $ax^2 + bx + c$ will factor is less than the chances that next January 4th you will be in St. Louis and will be hit by a thirty-pound fish dropped by a pigeon who used to be a pet of the youngest descendant of George Washington Carver.

Just at random I'll pick one: $3909929369948887x^2 + 7997923969696922 2005x + 5648713514644463$.
I bet that won't factor.

There are only a hundred or so "nice" quadratic polynomials that will factor. Every author of a beginning algebra book takes their examples from this small list.

In trig: Do you ever get an angle of exactly $\pi/6$ (= 60º) when you cut a pizza into six equal pieces? Never.

In calculus: Joe's favorite food is jelly beans. He throws one up in the air and catches it in his mouth as it falls. He throws it upward at a rate of 3 m/sec. One favorite calculus question is, "How high the jelly bean will go?"

One question in numerical analysis is, "Will the jelly bean ever be traveling at exactly 2 m/sec?"

Your Turn to Play

1. Well . . . will it ever travel at exactly 2 m/sec?
2. Let's see how much algebra you were taught. Almost everyone knows that you can solve any quadratic equation using the quadratic formula. Is there a formula for cubic equations (3rd degree), quartic equations (4th degree), quintic equations (5th degree) and so on?

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* Viz. is the abbreviation for *videlicet*. *Videlicet* is Latin for "that is to say" or "namely." Pronounced: wi-DAY-li-ket, where i is pronounced like the i in *if* or *big*.

.......COMPLETE SOLUTIONS.......

1. You may be surprised to find out that the answer is yes. There will be a time when the jelly bean is traveling at exactly 2 m/sec.

First of all, velocity of a jelly bean is a continuous function. There are no jumps in the graph. It can't be traveling at 0.5m/sec and an instant later be traveling at 0.4m/sec.

> **Happy Thought**
> Virtually all the math in numerical analysis is easier than the stuff in calculus.

Back in calculus, one of the main theorems was the Mean Value Theorem: If f is continuous on the interval [a, c] and differentiable on (a, c), then there is at least one point u in the interval in which

$$f'(u) = \frac{f(c) - f(a)}{c - a}$$

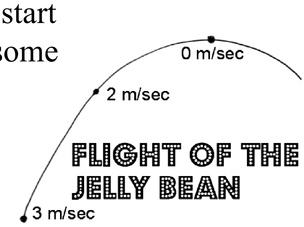
That was not easy to understand. Roughly translated, it says that if you are going from point a to point c and f(x) represents your location, then there has to be a point u on your trip where your velocity at u, f′(u), is equal to your average velocity over the whole trip, which is

$$\frac{f(c) - f(a)}{c - a}.$$

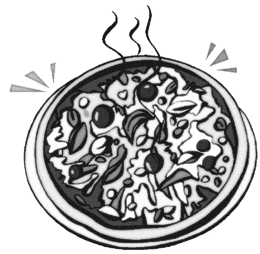
The much easier theorem is the Intermediate Value Theorem: If f is continuous on the interval [a, c], and v is any value between f(a) and f(c), then there must be a u in (a, c) such that f(u) = v.

Translation: If you are going from point a to point c, then every velocity between f(a) and f(c) must occur at least once.

Re-translation: If you are going 3 m/sec at the start of your trip and 0 m/sec later in your trip, then at some moment you must have been traveling at 2 m/sec.



Re-re-translation: The IVT (Intermediate Value Theorem) states that if you start to bake at pizza at 11 a.m. at 35º, and at noon it's 475º, then at some point between 11 and noon it will be exactly 243º.



In symbols: Given (11, 35º) and (12, 475º), then there exists a u such that 11 < u < 12 and (u, 243º).

The IVT was proved in Chapter 6 of *Life of Fred: Real Analysis.* But even this theorem is too complicated for numerical analysis. We are going to use a simplified version of the IVT.

2. One of the real delights of mathematics is the surprises that occasionally pop up.

Every beginning algebra student can solve any linear equation: first degree polynomials such as 89x – 12 = 57.

Every advanced algebra student can solve any quadratic equation: second degree polynomials such as $5x^2 - 46x + 7 = 0$, using the quadratic formula. The formula takes roots of sums of products of the coefficients.

Most mathematicians know of the cubic formula that can solve *any* cubic equation. It uses only roots and the arithmetic operations of addition, subtraction, multiplication, and division. Tartaglia found the formula in the early 1500s. It is complicated.

About 25 years later Ferrari found the quartic formula that can solve *any* 4^{th} degree equation.

Here is the summary so far . . .

1^{st} degree (linear)	We can solve any equation.
2^{nd} degree (quadratic)	We can solve any equation.
3^{rd} degree (cubic)	We can solve any equation.
4^{th} degree (quartic)	We can solve any equation.

Do you detect a pattern? Much of the work of mathematicians is finding patterns.

On the next page is the full chart for solving any polynomial equation.

Full Chart for Solving Polynomial Equations with Integer Coefficients

1st degree	We can solve any equation.
2nd degree	We can solve any equation.
3rd degree	We can solve any equation.
4th degree	We can solve any equation.
5th degree	No formula exists or will ever exist to solve every equation!
6th degree	No formula exists or will ever exist to solve every equation!
7th degree	No formula exists or will ever exist to solve every equation!
8th degree	No formula exists or will ever exist to solve every equation!
9th degree	No formula exists or will ever exist to solve every equation!
10th degree	No formula exists or will ever exist to solve every equation!
and so on.	

Why? I don't know. It's a mystery.

We can draw a line between any two points, but not every three.

2 is a magic number.

We can make three lines all mutually perpendicular to each other, but not four.

3 is a magic number.

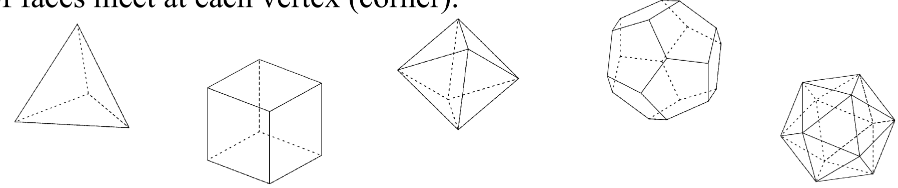
We can solve 1st, 2nd, 3rd, and 4th degree equations, but not 5th degree.

4 is a magic number.

We can find five regular Platonic solids, but not six.

5 is a magic number.

Platonic solids have faces that are all identical, and the same number of faces meet at each vertex (corner).



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